Event-triggered communication and H∞ control co-design for networked control systems

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Abstract

This paper studies an event-triggered communication scheme and an H_∞ control co-design method for networked control systems (NCSs) with communication delay and packet loss. First, an event-triggered communication scheme and a sampled-state-error dependent model for NCSs are presented. In this scheme and model, (a) the sensor takes samples in a periodic manner; (b) a triggering condition is applied to sampled signal to determine whether a signal is transmitted to the controller or not; and (c) the closed-loop system with a networked state feedback controller is modeled as a time-delay system. Secondly, by constructing a novel Lyapunov-Krasovskii functional, three theorems for the system asymptotical stability subject to imperfect communications are derived. Thirdly, a new algorithm is developed for the triggering condition and the controller feedback gain to meet the specified performance. This design algorithm is base on the two permissible limits on the signal transfer. These limits are: the maximum allowable communication delay bound and the maximum allowable number of successive packet losses, respectively. Finally, the proposed co-design method is demonstrated by two numerical examples.

Key words: Networked control systems, event-triggered communication scheme, co-design, packet loss.

1 Introduction

Feedback control systems using communication networks to close both information and control loops are called networked control systems (NCSs). They are becoming increasingly important in industrial process for many advantages. On the other hand, it is known that networked connection is not as reliable as traditional point-to-point connection. This has motivated a lot of interesting research, for example, see survey papers [6,9,25] and references therein. In the early study, some researchers investigated communication delay and packet loss as separate issues [11,21] while some others studied them together [4,22,15]. Recently, we have witnessed rapidly growing interest in NCSs. For example, various techniques have been proposed to deal with the above two issues [4,11,21]. Some new schemes are also proposed, such as event-triggered communication scheme [16,20], self-triggered control [1,2,17,19], deterministic or stochastic communication logic [23,26].

Most NCS schemes so far are based on time-triggered communications. In general, a time-triggered communication scheme leads to inefficient utilization of limited network resources. To mitigate the unnecessary waste of computation and communication resources in conventional time-triggered control, event-triggered control has been proposed [8,3,16,20]. In comparison to conventional time-triggered communication, event-triggering allows a considerable reduction of the network resource occupancy while maintaining the control performance. In [8,3,16,20], it is common to design a controller first based on an assumption that the signal transfer is perfect, and then to determine an event-triggering condition and/or network conditions to guarantee the stability and to maintain certain performance [3,16,20]. This can be considered as a two-step scheme. To the best of authors' knowledge, there is no result reported in the open literature on a co-design scheme, aimed at one design algorithm to achieve the desired performance while using less communication bandwidth. This motivates the re-

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search presented in this paper.

The two main contributions of this paper can be described as follows:

1) An event-triggered communication scheme, in which the sensor takes samples at a constant rate, but whether the last sampled-data should be transmitted or not is verified at every sampling instant by a special event-triggering condition. Furthermore, apart from the fact that this is part of an overall co-design method, other unique features are (a) not like the schemes in [16] and [20], where the event-triggering conditions need to be monitored continuously, here the condition is only checked at each sampling instance; and (b) it can be shown that the triggering condition proposed in this paper is in a general form, and that those conditions in [16,20] are special cases of this form.

2) Stability theorems and a co-design method. Different from some two-step schemes [3,16,20], controller gains and the event-triggering condition are designed in one step to meet H_∞ performance with respect to disturbance, and at the same time giving the maximum allowable communication delay bound (MADB) and the maximum allowable number of successive packet losses (MANSPL). An algorithm is developed for this co-design method and an initial analysis on its complexity is also presented.

This brief paper is organised in a sequence as described in the abstract. For ease of presentation, all proofs are presented in the Appendix.

2 Event-triggered communication scheme and NCSs modeling

In this section, we first propose an event-triggered communication scheme to reduce the number of transmitted data; then, we propose a control system model to link the event-triggered communication scheme with the other part of the system to be controlled; and finally a completed NCS model under a unified framework is presented.

To simplify the exposition, we make the following assumptions:

Assumption 1 *The sensors are time-triggered with a constant sampling period h. The sampling sequence is described by the set $\mathbb{S}_1 = \{0, h, 2h, \ldots, kh\}$, where $k \in \mathbb{N}$ and $k \to \infty$.*

Assumption 2 *Whether or not the sampled data should be transmitted over a communication network is determined by an event-triggered communication scheme. The transmission sequence at the sensors is described by the set $\mathbb{S}_2 = \{0, b_1h, b_2h, \ldots, b_kh\} \subseteq \mathbb{S}_1$, where $b_k \in \mathbb{N}$ and $\lim_{k\to\infty} b_k \to \infty$. Moreover, part of the data in $\mathbb{S}_2$ may be lost in the communication.*

Assumption 3 *The controllers and the actuators are event-triggered. The successfully transmitted sampled sequence at the sensors is described by the set $\mathbb{S}_3 = \{0, t_1h, t_2h, \ldots, t_kh\} \subseteq \mathbb{S}_2$, where $t_k \in \mathbb{N}$ and $\lim_{k\to\infty} t_k \to \infty$.*

Assumption 4 *The control input at the actuator is generated by a zero-order-holder (ZOH) with the holding time $t \in \Omega \triangleq [t_kh + \tau_{t_k}, t_{k+1}h + \tau_{t_{k+1}})$, where τ_{t_k} is the communication delay, h is the sampling period, and $t_kh + \tau_{t_k}$ are the instants when the control signal reaches the ZOH.*

Remark 1 *As special cases, if all sampled data are transmitted, we have $\mathbb{S}_1 = \mathbb{S}_2$, and the transmitter becomes time-triggered; If all broadcast release data are successfully transmitted, then we have $\mathbb{S}_3 = \mathbb{S}_2$. If $\mathbb{S}_2 \subset \mathbb{S}_1$, it means that there are some non-transmitted sampled signals. If $\mathbb{S}_3 \subset \mathbb{S}_2$, it means that there is packet loss in the transmission between the sensors and the controller.*

2.1 An event-triggered communication scheme

In this section, 2.1, we first consider a special case where there is communication delay in packet transmission but no packet loss. In section 3.2, we will study a general case where packet loss is also considered. Figure 1 shows a framework of the proposed event-triggered communication scheme for an NCS, where the transmitter has a logic function to determine whether or not the sampled data should be transmitted.

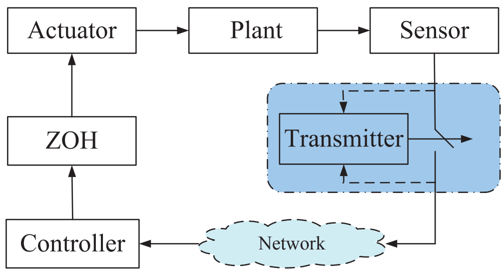

Fig. 1. A framework of an NCS with an event-triggering communication scheme

The transmission scheme described in Figure 1 is designed as

$$t_{k+1}h = t_kh + \min_{\ell}\{\ell h \,|\, e^T(i_kh)\Phi e(i_kh) \geq \delta_1 \wp\} \quad (1)$$

where $\wp = x^T(t_kh)\Phi x(t_kh)$, $\delta_1 \geq 0$ is a given scalar parameter, Φ is a positive definite weighting matrix to be designed, and $e(i_k h)$ is the error between the two states at the current sampling instant and the latest transmitted sampling instant, i.e. $e(i_k h) = x(i_k h) - x(t_k h)$, where $i_k h = t_k h + \ell h$, $\ell \in \mathbb{N}$; h is the sampling period of the sensor; t_k (k=1, 2, 3, ...) are some integers such that $\{t_1, t_2, t_3, \ldots\} \subset \{0, 1, 2, 3, \ldots\}$; $t_k h$ is the time at that instant a packet is successfully transmitted from the sensor.

Remark 2 *The communication scheme (1) is characterized by the parameters δ_1, Φ and h, which determine the communication load. How to design these parameters to reduce the amount of data transfer and at the same time to meet the performance requirement will be studied in Section 3. As a special case, if $\delta_1 = 0$ in (1), this leads to $t_{k+1}h = t_k h + h$, and this becomes a time-triggered sampling scheme in [15,24].*

2.2 NCSs modeling

Consider a class of linear systems governed by

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + B_\omega \omega(t) \\ z(t) = Cx(t) + Du(t) \end{cases} \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $\omega(t) \in \mathcal{L}_2[0, \infty)$ and $z(t) \in \mathbb{R}^p$ are state vector, control input vector, disturbance input vector, and controlled output vector respectively; A, B, B_ω, C and D are constant matrices with appropriate dimensions; the initial condition of the system (2) is given by $x(t_0) = x_0$. Throughout this paper, it is assumed that the system (2) is controlled over a communication network with a networked state feedback controller, which is directly connected to the actuator through a ZOH [27]. This leads to:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + B_\omega \omega(t) \\ z(t) = Cx(t) + Du(t) \\ u(t) = Kx(t_k h),\ t \in [t_k h + \tau_{t_k}, t_{k+1}h + \tau_{t_{k+1}}) \end{cases} \tag{3}$$

where K is the state feedback controller gain to be designed; τ_{t_k} is the packet transmission delay.

For a detailed timing analysis, we divide the holding interval of the ZOH $t \in \Omega$ into sampling-interval-like subsets $\Omega_\ell = [i_k h + \tau_{i_k}, i_k h + h + \tau_{i_k+1})$, i.e. $\Omega = \cup \Omega_\ell$, where $i_k h = t_k h + \ell h$, $\ell = 0, \ldots, t_{k+1} - t_k - 1$ means the sampling instants from the current transmitted sampling instant $t_k h$ to the future transmitted sampling instant $t_{k+1}h$; if ℓ takes the value of $t_{k+1} - t_k - 1$, then $\tau_{i_k+1} = \tau_{t_{k+1}}$, otherwise, $\tau_{i_k} = \tau_{t_k}$. See Fig.2 for an illustration. Define $\eta(t) \triangleq t - i_k h$, $t \in \Omega_\ell$. It is clear that $\eta(t)$ is a piecewise-linear function satisfying $0 < \eta_1 \leq \eta(t) \leq h + \bar{\tau} = \eta_3, t \in \Omega_\ell$ where $\eta_1 = inf_\ell\{\tau_{i_k}\}, \eta_3 = h + sup_\ell\{\tau_{i_k+1}\} = h + \bar{\tau}$; h and $\bar{\tau}$ are the sampling period

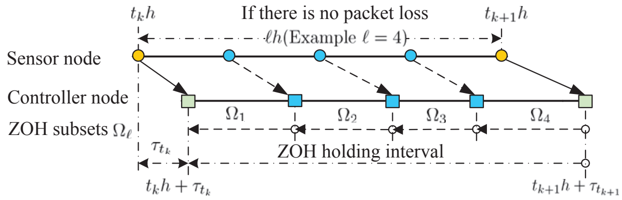

Fig. 2. Illustration of subsets of the ZOH

and the maximum allowable upper communication delay bound, respectively. Then, the control of (3) is:

$$u(t) = K(x(t - \eta(t)) - e(i_k h)), t \in \Omega_\ell \tag{4}$$

Combining (3) and (4) leads to a sampled-state-error dependent closed-loop NCS model

$$\begin{cases} \dot{x}(t) = Ax(t) + BK(x(t - \eta(t)) - e(i_k h)) + B_\omega \omega(t) \\ z(t) = Cx(t) + DK(x(t - \eta(t)) - e(i_k h)), t \in \Omega_\ell \end{cases} \tag{5}$$

where the initial condition of the state $x(t)$ is: $x(t) = \phi(t)$, $t \in [t_0 - \eta_3, t_0]$, $\phi(t_0) = x_0$, and $\phi(t)$ is a continuous function on $[t_0 - \eta_3, t_0]$.

The purpose of this paper is to provide an event-triggered communication and H_∞ control co-design method such that the system (5) is asymptotically stable with an H_∞ disturbance attenuation level γ, i.e., i) System (5) with $\omega(t) = 0$ is asymptotically stable; and ii) Under the zero initial condition, $\|z(t)\|_2 < \gamma \|\omega(t)\|_2$, for any nonzero $\omega(t) \in \mathcal{L}_2[0, \infty)$ and a prescribed $\gamma > 0$.

3 H_∞ stability analysis and controller design

In this section, we first develop two stability theorems for the system (5) with communication delay and with or without packet loss in the Sections 3.1 and 3.2, respectively. Then, Theorem 3 is presented in Section 3.3 which lays the foundation for the co-design algorithm presented in Section 3.4.

3.1 H_∞ stability analysis with communication delay but no packet loss

Theorem 1 *For some given positive constants η_1, η_3, γ and δ_1, a matrix K, under the event-triggered communication scheme (1), the system (5) is asymptotically stable with an H_∞ performance index γ for the disturbance attention, if there exist real matrices $P > 0$, $\Phi > 0$, $S > 0$, $R_j > 0$ (j=1, 2, 3), $\begin{bmatrix} Q_1 & * \\ Q_3 & Q_2 \end{bmatrix} > 0$, $\begin{bmatrix} R_i & * \\ U_i & R_i \end{bmatrix} > 0$ ($i = 2, 3$), matrices Q_3, U_2, U_3 with appropriate dimen-*

sions, such that

$$\begin{bmatrix} \Pi_{11}^i & * \\ \Pi_{21} & \Pi_{22} \end{bmatrix} < 0, i = 2, 3 \quad (6)$$

where $\Pi_{11}^i = [(1,1) = S - R_1,\ (2,1) = R_1,\ (2,2) = Q_1 - S - R_1 - R_2,\ (3,2) = (3-i)(R_2 - U_2),\ (3,3) = \delta_1\Phi - 2R_i + U_i^T + U_i,\ (4,2) = Q_3 + (3-i)U_2 + (i-2)R_2,\ (4,3) = R_i - (3-i)U_2 - (i-2)U_3^T,\ (4,4) = Q_2 - Q_1 - R_2 - R_3,\ (5,3) = (i-2)R_3 - U_3,\ (5,4) = (3-i)R_3 + (i-2)U_3 - Q_3,\ (5,5) = -Q_2 - R_3,\ (6,3) = -\delta_1\Phi,\ (6,6) = \delta_1\Phi - \Phi,\ (7,7) = -\gamma^2 I] + P\mathcal{I}_1 + \mathcal{I}_1^T P$ *and* $\Pi_{21} = col\{\eta_1 R_1\mathcal{I}_1,\ (\eta_2 - \eta_1)R_2\mathcal{I}_1,\ (\eta_3 - \eta_2)R_3\mathcal{I}_1,\ \mathcal{I}_2\}$, $\Pi_{22} = diag\{-R_1,\ -R_2,\ -R_3,\ -I\}$ *with* $\mathcal{I}_1 = [A,\ 0,\ BK,\ 0,\ 0,\ -BK,\ B_\omega], \mathcal{I}_2 = [C,\ 0,\ DK,\ 0,\ 0,\ -DK,\ 0]$.

Remark 3 *For a special case of a time-triggered scheme, i.e.,* $\delta_1 = 0$ *in (6), Theorem 1 will lead to less conservative results than those in some existing ones. For example, Consider the system* (2) *with the parameters:* $A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}$, *and* $K = -[3.75\ 11.5]$ *for a point-to-point non-networked connection. The MADB-s obtained based on the methods in [11], [27], [10] and [7] are 0.78, 0.86, 0.94 and 1.04, respectively. It can be shown that applying Theorem 1, one can get a better result of MADB=1.07.*

3.2 Stability analysis with communication delay and packet loss

With the communication scheme (1), if there is no packet loss, the transmitter can directly utilize the violation of (1) to send packets while guaranteeing the H_∞ stability of the system (5). With packet loss, (1) cannot be directly employed to determine if the sampled data should be transmitted. Therefore, (1) is changed to:

$$b_{k+1}h = b_k h + \min_{\nu}\{\nu h \left| \tilde{e}^T(l_k h)\Phi\tilde{e}(l_k h) > \delta_2\Im \right.\} \quad (7)$$

where $\Im = x^T(b_k h)\Phi x(b_k h)$, $\tilde{e}(l_k h) = x(l_k h) - x(b_k h)$, $l_k h = b_k h + \nu h$, $\nu \in \mathbb{N}$, $\delta_2 > 0$ is a given scalar parameter, $\Phi > 0$ as defined in (1), $b_k h$ is the transmitted sampling instant as defined in Assumption 2.

Remark 4 *Notice that (i) the state error and* $l_k h$ *in (7) are different from those in (1); (ii)* δ_2 *in (7) should be less than or equal to* δ_1 *in (1) to take into account the extra communication delay caused by the packet loss.*

Next a new theorem is developed for achieving the H_∞ performance while there is communication delay and packet loss in the signal transfer.

Theorem 2 *For some given positive constants* η_1, η_3, γ, h, δ_i $(i = 1, 2, 3)$, *a matrix* K, *under the event-triggered communication scheme (7), the system (5) is asymptotically stable with an* H_∞ *performance index* γ *for the disturbance attention, if there exist real matrices* $P > 0$, $\Phi > 0$, $S > 0$, $R_j > 0$ $(j = 1,\ 2,\ 3)$, $\begin{bmatrix} Q_1 & * \\ Q_3 & Q_2 \end{bmatrix} > 0$, $\begin{bmatrix} R_i & * \\ U_i & R_i \end{bmatrix} > 0$ $(i = 2,\ 3)$, *matrices* Q_3, U_2, U_3 *with appropriate dimensions, such that (6) holds, and the number of successive packet losses* d_k *satisfies*

$$d_k \le d_{MANSPL} \triangleq \left\lfloor \log_{(1+\sqrt{\delta_2})(1+\varepsilon)} \frac{1+\sqrt{\delta_1}}{1+\sqrt{\delta_2}} \right\rfloor \quad (8)$$

where $\varepsilon = \left|e^{Ah} - I\right| + \left|\frac{he^{Ah}BK}{1-\sqrt{\delta_1}}\right| + \left|\delta_3 e^{Ah}B_\omega\right|$ *and* $\lfloor\diamond\rfloor$ *gives the largest integer smaller than or equal to* $\diamond$.

Remark 5 *The maximum allowable number of successive packet losses* d_{MANSPL} *in (8) is a non-negative integer. This implies that* $\delta_2 \le \delta_1$. *As a special case, when* $\delta_2 = \delta_1$, $d_{MANSPL} = 0$, *i.e. packet loss is not permitted.* $\delta_2 \le \delta_1$ *also means to lower the threshold of the triggering condition which causes more packets being transmitted. This is necessary since* δ_1 *in (1) assumes no packet loss.*

3.3 H_∞ Controller design

The next theorem lays the foundation for the co-design algorithm presented in the next section.

Theorem 3 *For some given positive constants* η_1, η_3, h, γ, δ_i $(i = 1, 2, 3)$, *under the given communication scheme (7), the system (5) is asymptotically stable with an* H_∞ *performance index* γ *for disturbance attention and a state feedback gain* $K = YX^{-1}$, *if the number of successive packet losses* d_k *satisfies (8) and there exist real matrices* $X > 0$, $\tilde{S} > 0$, $\tilde{\Phi} > 0$, $\tilde{R}_j > 0$ $(j = 1,\ 2,\ 3)$, $\begin{bmatrix} \tilde{Q}_1 & * \\ \tilde{Q}_3 & \tilde{Q}_2 \end{bmatrix} > 0$, $\begin{bmatrix} \tilde{R}_i & * \\ \tilde{U}_i & \tilde{R}_i \end{bmatrix} > 0$ $(i = 2,\ 3)$, *matrices* $\tilde{Q}_3$, $\tilde{U}_2$, $\tilde{U}_3$, Y *with appropriate dimensions, such that*

$$\begin{bmatrix} \tilde{\Pi}_{11}^i & * \\ \tilde{\Pi}_{21} & \tilde{\Pi}_{22} \end{bmatrix} < 0, i = 2, 3 \quad (9)$$

where $\tilde{\Pi}_{11}^i = [(1,1) = \tilde{S} - \tilde{R}_1,\ (2,1) = \tilde{R}_1,\ (2,2) = \tilde{Q}_1 - \tilde{S} - \tilde{R}_1 - \tilde{R}_2,\ (3,2) = (3-i)(\tilde{R}_2 - \tilde{U}_2),\ (3,3) = \delta_1\tilde{\Phi} + \tilde{U}_i^T + \tilde{U}_i - 2\tilde{R}_i,\ (4,2) = \tilde{Q}_3 + (3-i)\tilde{U}_2 + (i-2)\tilde{R}_2,\ (4,3) = \tilde{R}_i - (3-i)\tilde{U}_2 - (i-2)\tilde{U}_3^T,\ (4,4) = \tilde{Q}_2 - \tilde{Q}_1 - \tilde{R}_2 - \tilde{R}_3,\ (5,3) = (i-2)\tilde{R}_3 - \tilde{U}_3,\ (5,4) = (3-i)\tilde{R}_3 + (i-2)\tilde{U}_3 - \tilde{Q}_3,\ (5,5) = -\tilde{Q}_2 - \tilde{R}_3,\ (6,3) = -\delta_1\tilde{\Phi},\ (6,6) = \delta_1\tilde{\Phi} - \tilde{\Phi},\ (7,7) = -\gamma^2 I] + \tilde{\mathcal{I}}_1 + \tilde{\mathcal{I}}_1^T$, *and* $\tilde{\Pi}_{21} = col\{\eta_1\tilde{\mathcal{I}}_1,\ (\eta_2 - \eta_1)\tilde{\mathcal{I}}_1,\ (\eta_3 - \eta_2)\tilde{\mathcal{I}}_1,\ \tilde{\mathcal{I}}_2\}, \tilde{\Pi}_{22} = -diag\{X\tilde{R}_1^{-1}X,$

$X\tilde{R}_2^{-1}X, X\tilde{R}_3^{-1}X, I\}$ *with* $\tilde{\mathcal{I}}_1 = [AX, 0, BY, 0, 0, -BY, B_\omega], \tilde{\mathcal{I}}_2 = [CX, 0, DY, 0, 0, -DY, 0]$.

3.4 Communication and control co-design algorithm

The parameters δ_1, δ_2, Φ in the event-triggering scheme and the controller gain K are coupled together in Theorem 3, and at the same time the control performance and the network resource usage are related to these parameters. Thus, it is necessary to develop an algorithm to obtain these parameters simultaneously for the desired H_∞ performance while using less network resource.

For convenience, the average transmission time $\tilde{T}$ is defined as the ratio of a given period of time T to the number of transmitted sampled-data.

Algorithm 4: Find the communication parameters δ_1, δ_2, Φ and the controller gain K

Step 1. For the given d_{MANSPL} and $\bar{\tau}$, set $\delta_1 = \delta_1 + \lambda$, where λ is the step increment of δ_1, δ_1 is initially set to zero, $\delta_1 \in [0, 1)$.

Step 2. For a given δ_1 in Step 1, if there exists a feasible solution satisfying LMIs defined in the CCL algorithm, go to Step 3; Otherwise go to Step 1.

Step 3. Use the Matlab LMI Toolbox and the CCL algorithm to find the maximum $\eta_3(\delta_1)$, and the corresponding $\Phi(\delta_1)$, $K(\delta_1)$ based on Theorem 3.

Step 4. Use the given d_{MANSPL} and the current δ_1 to calculate the maximum $\delta_2(\delta_1, d_{MANSPL})$ under the constraint of (8). If $\delta_2(\delta_1, d_{MANSPL}) \leq 0$, go to Step 1.

Step 5. Set the sampling period $h = \eta_3(\delta_1) - \bar{\tau}$. If $h \leq 0$, go to Step 1; Else based on $\Phi(\delta_1)$ and $K(\delta_1)$ in Step 3, and $\delta_2(\delta_1, d_{MANSPL})$ in Step 4, set a simulation time T, and based on the communication scheme (7), use Matlab/Simulink to find the average transmission time $\tilde{T}$.

Step 6. Go to Step 1 for another value of δ_1, if feasible, find another $\tilde{T}$ for this particular δ_1, until $\delta_1 \geq 1$ when the search is terminated.

Notice that, in the mathematical theorems, for a given set of parameters and the performance requirement, one can find d_{MANSPL} and $\bar{\tau}$; whereas in the design process, one decide d_{MANSPL} and $\bar{\tau}$ first based on the knowledge of the network being used, then find the parameters using the above algorithm. In addition, since $\delta_1 = \delta_1 + \lambda$ and $\delta_1 \in [0, 1)$ in Step 1, Algorithm 4 terminates in a finite number of steps $M \in \mathbb{N}^+$ and $M \leq 1/\lambda$.

The computational complexity of Algorithm 4 can be estimated from: (a) the number of scalar decision variables N in the LMI, (b) the number of LMI rows L, and (c) λ: the step increment of δ_1. Based on the interior point methods used by the Matlab/LMI Control Toolbox, the complexity of Algorithm 4 can be estimated as being proportional to $N^3 L \lambda^{-1}$ [13]. In the case studied in this paper, since $N = 11.5n^2 + 2.5n$ and $L = 22n$ fixed for a given n, where n is the dimension of state variable given in (2), the computational complexity and the searching precision are related to λ directly. For a small λ, it requires more computations but gives more search results.

4 Illustrative Examples

This section uses two examples to demonstrate the effectiveness of the proposed design method.

Example 1: Consider the same example as that in [3,12]:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{10}$$

To compare like to like, (a) set $\Phi = I$, (b) ε in [3] equals to zero, (c) choose $K = [1, -4]$, and (d) assuming that there is no communication delay. In this special case, Table 1 lists the minimum inter-event times for different δ_1 values of obtained by the methods [3], [12], and the one in this paper based on Theorem 1, respectively. Notice that, to guarantee the desired performance, for four cases in Table 1, the method in [3], [12] fail to give solutions.

Table 1
Lower bound of the inter-event times and allowable upper bound of δ_1 (Example 1)

δ_1	0.003	0.0273	0.0588	0.10
[12]	0.0318	Fail	Fail	Fail
[3]	–	0.0840	0.1136	Fail
Theorem 1	0.2141	0.1922	0.1602	0.1125

Example 2: Consider an inverted pendulum on a cart controlled over a network, the linearized plant model (2) with the parameters [18]:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{m_1 g}{-m_2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{l} & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{1}{m_2} \\ 0 \\ \frac{-1}{m_2 l} \end{bmatrix}, B_\omega = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{11}$$

In this simulation, $m_1 = 1$, $m_2 = 10$, $l = 3$, and $g = 10$, and the initial state is x_0 =[0.98, 0, 0.2, 0]^T; for comparison, set $\gamma = 200$, $\bar{\tau} = 0$ and $\omega(t) = 0.01\sin(2\pi t)$.

First consider the case of no packet loss, set $\lambda = 0.1$, $h = 0.01s$ and the simulation time $T = 30s$. Based on

Algorithm 4, Table 2 lists the maximum allowable η_3, the controller gain K, and the average transmission period $\tilde{T}$. $\tilde{T}$ obtained in this paper is much larger than $0.08s$ in [18] and $0.23s$ in [19]. Compare with the simple time-triggered scheme in [15,24], for example, on average only 22%, 14% sampled data are transmitted when $\delta_1 = 0.3$ and 0.5, respectively.

Table 2
Maximum allowable η_3 and obtained controller gain with different δ_1 (Example 2)

δ_1	η_3	$\tilde{T}$	$\eta_3/\tilde{T}$	K
0.0	0.18	0.18	100%	[10.88 34.99 437.50 248.70]
0.1	0.13	0.43	31%	[12.02 33.64 447.60 254.77]
0.3	0.09	0.42	22%	[10.49 28.13 411.72 234.21]
0.5	0.05	0.38	14%	[16.22 41.08 516.03 294.91]

Next for the case of packet loss in the communication, set $\lambda = 0.2$, $h = 0.01s$, $\delta_3 = 0.01$ and the simulation time $T = 30s$. For different values of δ_1 and δ_2, Table 3 lists the d_{MANSPL} and the corresponding $\tilde{T}$ based on Algorithm 4. It can be seen that the proposed event-triggered scheme allows some successive packet losses. In particular, for given d_{MANSPL} and η_3, one can choose the parameters in the event-triggering scheme and the controller gain from Tables 2 and 3. For example, if $d_{MANSPL} \leq 2$ and $\eta_3 \leq 0.05$, one may choose $\delta_1 = 0.5$, $\delta_2 = 0.02$, and $K = [16.22, 41.08, 516.03, 294.91]$.

Table 3
MANSPL and average transmission period $\tilde{T}$ with different δ_1, δ_2 (Example 2)

δ_1	δ_2	η_3	$\tilde{T}$	d_{MANSPL}	$\eta_3/\tilde{T}$
0.7	0.02	0.03	0.12	3	25%
0.7	0.04	0.03	0.16	2	18%
0.5	0.02	0.05	0.16	2	31%
0.5	0.04	0.05	0.20	1	25%
0.3	0.02	0.09	0.32	2	28%
0.3	0.04	0.09	0.34	1	26%
0.1	0.02	0.13	0.30	1	43%

When $\delta_1 = 0.7$, $\delta_2 = 0.02$ and $h = 0.01s$, the release times, the packet loss and the system states are shown in Fig. 3, and the average transmission intervals at $2s$, $4s, \ldots,$ $16s$ are shown in Fig. 4. This shows that: (a) despite packet loss, the system remains asymptotically stable; and (b) the average transmission intervals fluctuate in a small range when the system state approaches the operating point.

5 Conclusion

A combined event-triggering-condition and controller-feedback-gain co-design method for networked control systems is presented in this paper. This design method with the proposed algorithm maintains the desired system H_∞ performance, takes into account communication delay and packet loss in the networked signal transfer, and makes a better use of network resources. The theoretical background of the proposed design method is a novel Lyapunov-Krasovskii functional and the three theorems proved in this paper. The application of the design method and some of its advantages over other existing methods are demonstrated by two numerical examples.

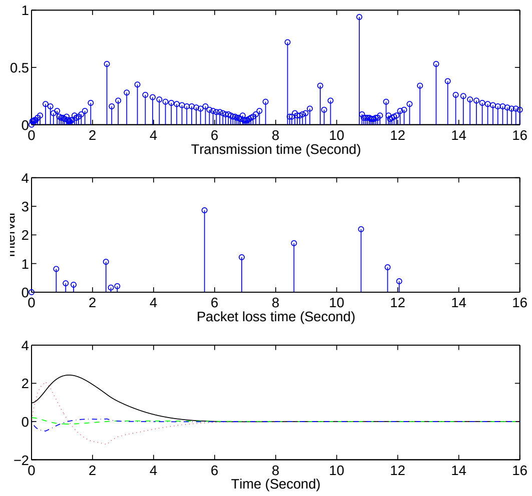

Fig. 3. Communication condition, packet loss and state responses (Example 3)

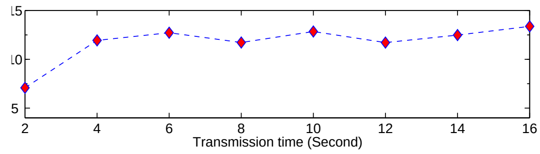

Fig. 4. Average transmission interval (Example 3)

Appendix

Proof (Proof of Theorem 1): Construct a Lyapunov-Krasovskii functional candidate as

$$V(t, x_t) = x^T(t)Px(t) + \int_{t-\eta_1}^{t} x^T(v)Sx(v)dv + \sum_{i=1}^{3}(\eta_i - \eta_{i-1}) \int_{-\eta_i}^{-\eta_{i-1}} \int_{t+s}^{t} \dot{x}^T(v)R_i\dot{x}(v)dvds + \int_{t-\rho}^{t} \xi^T(v)Q\xi(v)dv, t \in \Omega_\ell \quad (12)$$

where $P > 0$, $S > 0$, $R_i > 0$ $(i = 1, 2, 3)$, $\eta_0 = 0$, $\rho \triangleq \frac{\eta_3-\eta_1}{2}$, $\eta_2 \triangleq \frac{\eta_1+\eta_3}{2}$ and $Q \triangleq \begin{bmatrix} Q_1 & * \\ Q_3 & Q_2 \end{bmatrix} > 0, \xi(v) \triangleq \begin{bmatrix} x(v-\eta_1) \\ x(v-\eta_2) \end{bmatrix}$.

Taking the time derivative of $V(t, x_t)$ with respect to t along the trajectory of (5) yields

$$\begin{aligned}\dot{V}(t,x_t) = & 2x^T(t)P\dot{x}(t) - x^T(t-\eta_1)Sx(t-\eta_1) \\ & +\xi^T(t)Q\xi(t) - \xi^T(t-\rho)Q\xi(t-\rho) \\ & +x^T(t)Sx(t) + \sum_{i=1}^{3} \dot{x}^T(t)(\eta_i - \eta_{i-1})^2 R_i\dot{x}(t) \\ & -\sum_{i=1}^{3}(\eta_i - \eta_{i-1})\int_{t-\eta_i}^{t-\eta_{i-1}} \dot{x}^T(v)R_i\dot{x}(v)dv \quad (13)\end{aligned}$$

From the communication scheme (1), for $i_k h \in [t_k h, t_{k+1}h)$, it is clear that

$$e^T(i_k h)\Phi e(i_k h) \leq \delta_1 x^T(t_k h)\Phi x(t_k h) \quad (14)$$

Applying Jensen's inequality [5] and reciprocally convex approach [14] to deal with the integral items in (13):

$$\dot{V}(t,x_t) \leq \varrho^T(t)\Pi\varrho(t) - z^T(t)z(t) + \gamma^2\omega^T(t)\omega(t) \quad (15)$$

where $\Pi = \Pi_{11}^i - \Pi_{21}^T\Pi_{22}^{-1}\Pi_{21}, \Pi_{11}^i, i = 2, 3, \Pi_{21}$ and Π_{22} being defined in Theorem 1 and $\varrho^T(t) = [x^T(t), x^T(t-\eta_1), x^T(t-\eta(t)), x^T(t-\eta_2), x^T(t-\eta_3), e^T(i_k h), \omega^T(t)]$.

Using the Lyapunov-Krasovskii functional (12), and from (6) and (15), one can derive that the system (5) with $\omega(t) = 0$ is asymptotically stable and $\|z(t)\|_2 < \gamma\|w(t)\|_2$ under the zero initial condition. This completes the proof. □

Proof (Proof of Theorem 2): Consider a successful broadcast release interval $[t_k h, t_{k+1}h)$, and assume that in this interval, there are d_k unsuccessfully transmitted broadcast packets: $t_k = b_0 < b_1 < b_2 < \ldots < b_{d_k} < b_{d_k+1} = t_{k+1}$. For $l = 0, 1, \ldots, d_k$, applying the communication scheme (7) yields

$$|x(b_{l+1}h - h) - x(b_l h)| \leq \sqrt{\delta_2}\,|x(b_l h)| \quad (16)$$

From (16):

$$\begin{aligned}|x(b_{l+1}h - h)| &\leq |x(b_{l+1}h - h) - x(b_l h)| + |x(b_l h)| \\ &\leq (1 + \sqrt{\delta_2})\,|x(b_l h)| \quad (17)\end{aligned}$$

Since

$$\begin{aligned}|x(t_k h)| &= |x(b_{l+1}h - h) - x(b_{l+1}h - h) + x(t_k h)| \\ &\leq |x(b_{l+1}h - h)| + \sqrt{\delta_1}\,|x(t_k h)|, \quad (18)\end{aligned}$$

one can get

$$(1 - \sqrt{\delta_1})\,|x(t_k h)| \leq |x(b_{l+1}h - h)| \quad (19)$$

For the closed-loop system (5) with a disturbance input vector $\omega(t) \in \mathcal{L}_2[0,\infty)$, assume that there exists a positive real constant $\delta_3 > 0$ such that $\left|\int_t^{t+h}\omega(\tau)d\tau\right| \leq \delta_3\,|x(t)|$ for all $t \geq 0$. Then based on the solution $x(t)\,|_{t=b_{l+1}h}$ of (3) and (19):

$$\begin{aligned}&|x(b_{l+1}h) - x(b_{l+1}h - h)| \\ &\leq \left|e^{Ah} - I\right|\,|x(b_{l+1}h - h)| \\ &\quad + \left|he^{Ah}BKx(t_k h)\right| + \left|e^{Ah}B_\omega\int_{b_{l+1}h-h}^{b_{l+1}h}\omega(\tau)d\tau\right| \\ &\leq \varepsilon\,|x(b_{l+1}h - h)| \quad (20)\end{aligned}$$

where $\varepsilon > 0$ is defined in Th.2.

From (17) and (20):

$$|x(b_{l+1}h) - x(b_{l+1}h - h)| \leq \varepsilon(1 + \sqrt{\delta_2})\,|x(b_l h)| \quad (21)$$

Considering (16) and (21) together, for $t \in [b_{d_k}h, b_{d_k+1}h)$, the state error between t and $t_k h$ is

$$\begin{aligned}&|x(t) - x(t_k h)| \\ &\leq |x(t) - x(b_{d_k}h)| + \left|\sum_{l=0}^{d_k-1}(x(b_{l+1}h - h) - x(b_l h))\right| \\ &\quad + \left|\sum_{l=0}^{d_k-1}(x(b_{l+1}h) - x(b_{l+1}h - h))\right| \\ &\leq \sum_{l=0}^{d_k}\sqrt{\delta_2}\,|x(b_l h)| + \sum_{l=0}^{d_k-1}\varepsilon(1 + \sqrt{\delta_2})\,|x(b_l h)| \quad (22)\end{aligned}$$

From (16) and (21):

$$\begin{aligned}&|x(b_{l+1}h)| \\ &\leq |x(b_{l+1}h) - x(b_{l+1}h - h)| \\ &\quad + |x(b_{l+1}h - h) - x(b_l h)| + |x(b_l h)| \\ &\leq [(1 + \varepsilon)(1 + \sqrt{\delta_2})]^{l+1}\,|x(t_k h)| \quad (23)\end{aligned}$$

Using (23) to deal with the term of $|x(b_l h)|$ in (22), yields

$$|x(t) - x(t_k h)|$$

$$\le \sum_{l=0}^{d_k} \sqrt{\delta_2}\,|x(b_l h)| + \sum_{l=0}^{d_k-1} \varepsilon(1+\sqrt{\delta_2})\,|x(b_l h)|$$
$$\le ((1+\sqrt{\delta_2})^{d_k+1}(1+\varepsilon)^{d_k} - 1)\,|x(t_k h)| \quad (24)$$

Considering (8) and (24) together leads to

$$|x(t) - x(t_k h)| \le \sqrt{\delta_1}\,|x(t_k h)| \quad (25)$$

From (25), one can see that the communication condition (1) in Theorem 1 is ensured by the communication scheme (7), this reveals that Theorem 2 can be readily derived from Theorem 1 if the communication scheme (7) is applied. This completes the proof. □

Proof (Proof of Theorem 3): Define $X = P^{-1}$, $X\Phi X = \tilde{\Phi}$, $XSX = \tilde{S}$, $XR_jX = \tilde{R}_j$, $XQ_jX = Q_j$, $j = 1, 2, 3$, $XU_iX = \tilde{U}_i$, $i = 2, 3$, and $Y = KX$; pre- and post-multiply both sides of leftmost matrix of (6) with diag$(X, X, X, X, X, X, I, I, I, I)$ and its transpose, respectively; (9) can be reality obtained from Theorem 2. This completes the proof. □

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